Barycentric Coordinate and its Applications

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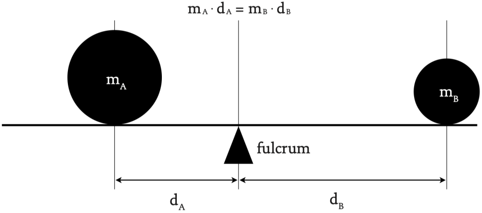
Overview

Introduction:

To understand Barycentric Coordinate, we need to introduce some basic terms and theorems.

- Archimedes' law of lever tells us that "two magnitudes balance at distances reciprocally proportional to their magnitudes" [2]

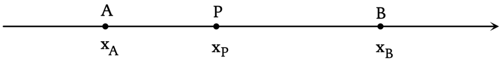


[1]

- If we denote the three vertices of a 2D triangle as $A = (x_a, y_a)$, $B = (x_b, y_b)$, $C = (x_c, y_c)$, then the edge AB can be defined as $B - A$, and edge AC as $C - A$.

Definition:

Let's first consider the Barycenter P on a real line.



[1]

- with the mass on A as m_A and on B as m_B. Following the Archimedes' Law of the lever, we have $m_A\left(x_P - x_A\right) = m_B\left(x_B - x_P\right)$.
- By solving the equation, we get $x_P = w_A x_A + w_B x_B$ where $w_A = \frac{m_A}{m_A + m_B}$ and $w_B = \frac{m_B}{m_A + m_B}$
- Here, x_P is our Barycenter P

Definition:

Now, let's move the idea to triangle with vertices A, B, C

- We can write the Barycentric Coordinate P as a polynomial about A, B, C with coefficients β and γ.
- $p = A + \beta(B - A) + \gamma(C - A) \quad = A + \beta B - \beta A + \gamma C - \gamma A$
  $= (1 - \beta - \gamma)A + \beta B + \gamma C$
- Therefore, the Barycentric Coordinate P is defined as:

$$P = \alpha A + \beta B + \gamma C$$

where $\alpha + \beta + \gamma = 1$

Properties:

There are some interesting properties about the Barycentric Coordinate P of a triangle.

- A point P is inside the ABC triangle if and only if:

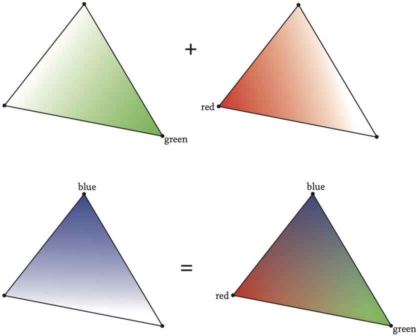
$$0 < \alpha < 1$$
$$0 < \beta < 1$$
$$0 < \gamma < 1$$

- By changing the values of α, β, γ between 0 and 1, the point P will move smoothly inside the triangle.
- The values of α, β, γ corresponds to the quotient of the areas of triangles ABP, ACP, BCP to the area of whole triangle ABC:

$$\alpha = S_{BPC}/S_{ABC}$$
$$\beta = S_{APC}/S_{ABC}$$
$$\gamma = S_{APB}/S_{ABC}$$

Applications:

- The most common application of Barycentric Coordinate is shading the triangle with RGB coloring.



[1]

- Take the green coloring for example. We set the weight of the right corner vertex to 256, and the rest two vertices to 0. When we toggle the values of α, β, γ, the color will change as the coordinates of P change.

LBC:

- One innovative use of Barycentric Coordinates is applying it on the deformation of the graph objects. The method uses the LBC (Local Barycentric Coordinates)[3].
- We first define $c_1, c_2, ..., c_n$ to be the control points which are the vertices of a closed control cage. And let $\mathbb{S}$ be the domain bounded by the cage. The goal is to find a function $w_i : \mathbb{S} \mapsto \mathbb{R}$ for each c_i, such that $[w_1(x), \ldots, w_n(x)]$ is a set of generalized barycentric coordinates of $x \in \mathbb{S}$ with respect to the control points $\{c_i\}$.
- These coordinate functions are used for interpolating function values $f(c_1) \ldots, f(c_n)$ defined as $f(x) = \sum_{i=1}^{n} w_i(x) f(c_i)$ which is used for shape deformation.

LBC:

- The most important thing about LBC is that it fulfills the locality: a control point only influences its nearby regions, and a point $x \in \mathbb{S}$ is influenced by a small number of control points.
- To calculate LBC, we aim to minimize a target function based on Total Variation. So we can write it out as a convex optimization problem[3]:

$$\begin{aligned} &\min_{w_1,\ldots,w_n} \sum_{i=1}^{n} \int_{\Omega} |\nabla w_i| \\ \text{s.t. } &\sum_{i=1}^{n} w_i(x) c_i = x, \sum_{i=1}^{n} w_i = 1, w_i \geq 0, \forall x \in \Omega \\ &w_i\left(c_j\right) = \delta_{ij} \forall i, j \end{aligned}$$

Results:

- The results of image deformation are shown below:

[Original Figure] [3]

[After Deformation] [3]

Reference

P. Lidberg, *Barycentric and wachspress coordinates in two dimensions: Theory and implementation for shape transformations*, 2011.

P. Strathern, *Archimedes the fulcrum: the big idea*, Arrow Books, 2010.

J. Zhang, B. Deng, Z. Liu, G. Patanè, S. Bouaziz, K. Hormann, and L. Liu, *Local barycentric coordinates*, ACM Trans. Graph., 33 (2014).

Thank You